NONLINEAR ROBUSTNESS ANALYSIS VIA PIECEWISE CONSTANT LYAPUNOV FUNCTIONS

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Abstract. The contribution treats the estimation of a robust region of attraction for nonlinear closed loop systems with structured uncertainties. For this goal a piecewise constant parameter dependent Lyapunov function is constructed. The latter is based on dividing the uncertainty bounding set into a finite number of partitions, yielding a common Lyapunov function for each partition. A numerical example is provided.

Key Words. Nonlinear uncertain systems, piecewise constant Lyapunov function, robust region of attraction.

1. INTRODUCTION

There are many basic issues concerned with the problem of stabilizing nonlinear systems with parametric uncertainty (e. g. see [13] and [10]). Relatively little attention has been devoted to the estimation of a robust region of attraction for such systems in the closed-loop situation. Although such an analysis may be very useful in many practical applications. This contribution addresses an uncertain time invariant nonlinear systems $\dot{x} = A(q)\,x + f(q,\,x)$ with a locally stable equilibrium at 0, where a robust region of attraction G_Ω is to be determined: G_Ω is requested to be a neighborhood of 0, such that any solution with initial state $x(0) \in G_\Omega$ converges to 0 for every fixed q from the set of parameters Ω. The problem is treated in literature by both approaches via common Lyapunov functions (see [3], [12] and [11]) and parameter dependent Lyapunov functions (see [2], [8], [1] and [5]). Often a parameter dependent Lyapunov function is advantageous since it happens that an uncertain system is robustly stable while no common Lyapunov function exists (see the example in [9]). However, our approach settles in between these two methods, proposing piecewise constant parameter dependent Lyapunov functions to determine a robust region of attraction. For this goal a continuous parameter dependent Lyapunov function is approximated by a piecewise constant parameter dependent Lyapunov function, determined from the system's linearization A. Therefore the considered radius formula for a region of attraction has to be evaluated for a finite number of parameter values only. This results in the proposed method being implemented as a computer program in a straight forward manner.

2. SYSTEM CLASS

We start of from the linearization about a known equilibrium of an uncertain time invariant system: Let

$$A\colon \mathbb{R}^m \to \mathbb{R}^{n\times n}\,, \quad f\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n\,. \tag{1}$$

The system then is represented by the parameter dependent differential equation

$$\dot{x}(t) = A(q)\,x(t) + f(q,\,x(t))\,. \tag{2}$$

Here q denotes the *uncertain parameter*, only known to be valued within an *uncertainty bounding set* $\Omega \subset \mathbb{R}^m$. For the scope of this paper the following restrictions are made on the system class:

(A1) $f(q_0,\;\cdot\;)$ is locally Lipschitz for every fixed $q_0 \in \Omega$ (i. e. the Lipschitz constant may depend on q_0).

(A2) For every bounded subset Ω^- of Ω and every $\varepsilon > 0$ there exists a $\delta(\Omega^-,\,\varepsilon) > 0$ such that the following condition holds for all $q_0 \in \Omega^-$ and all $\xi \in \mathbb{R}^n,\ \xi \neq 0,\ ||\xi||_2 < \delta(\Omega^-,\,\varepsilon)$:

$$||f(q_0,\,\xi)||_2\,||\xi||_2^{-1} < \varepsilon\,.$$

Here $||\cdot||_2$ denotes the Euclidean vector norm.

(A3) A is an affine map, i. e. the map

$$H\colon \mathbb{R}^m \to \mathbb{R}^{n\times n},\;\; \eta \mapsto A(\eta) - A(0)\;,$$

is linear.

(A4) $A(q_0)$ is stable for all $q_0 \in \Omega$. That is

$$\sigma(A(q_0)) \subset \mathbb{C}^- \quad \forall\, q_0 \in \Omega\;,$$

where $\sigma(A(q_0))$ denotes the set of eigenvalues of $A(q_0)$ and $\mathbb{C}^-$ is the open left half plane of $\mathbb{C}$.

(A5) Ω is compact.

Roughly speaking, (A2) demands f to contain no linear terms. Indeed for subsets $\Omega^- = \{q_0\}$ holding only one parameter value $q_0 \in \Omega$, assumption (A2) is fulfilled when A and f are established by standard linearization about an equilibrium point. Then the restrictive aspect of (A2) is that we demand uniformity w. r. t. the parameter. Such a situation is given when a suitable $\delta(\{q_0\}, \varepsilon)$ can be established, depending continuously on q_0 for all fixed $\varepsilon > 0$. Furthermore (A2) is *not only* meant to be an assumption of *existence*: In order to apply the proposed method $\delta(\Omega^-, \varepsilon)$ has to be *known* for the case that Ω^- is a cube.

Note that (A3) also covers the situation when the linear term is a so called interval matrix. Therefore set $m = n^2$ and define Ω to be the product of the desired intervals. Then let $A(q_0)$, $q_0 \in \Omega$, denote the matrix holding the entries of the vector q after a suitable rearrangement. This clearly yields an affine map A.

We state some basic properties immediately implied by (A1) to (A5):

(P1) For every fixed $q \in \Omega$ the right hand side of the differential equation (2) is locally Lipschitz. Hence for every pair $(q,\, x_0) \in \Omega \times \mathbb{R}^n$ the equation (2) together with the initial condition $x(0) = x_0$ has an unique solution in some neighborhood of x_0. Let

$$T^+(q,\, x_0) = [0,\, t^+(q,\, x_0)) \subseteq \mathbb{R}_0^+\;,\\ t^+(q,\, x_0) \in \mathbb{R}_0^+ \cup \{\infty\}$$

denote the maximal interval on the time axis, to which the solution can be extended uniquely. Further let

$$\varphi(q,\, x_0,\; \cdot\;)\colon T^+ \to \mathbb{R}^n$$

denote the solution itself.

(P2) $A(\eta) = A(\eta_0) + H(\eta - \eta_0)$ for all $\eta,\ \eta_0$ in Ω.

(P3) For every fixed $q_0 \in \Omega$ the Lyapunov equation $A(q_0)^\top P_0 + P_0\, A(q_0) = -I\;,\; P_0 > 0$ has an unique solution $P_0 \in \mathbb{R}^{n\times n}$. Therefore the map $P\colon \Omega \to \mathbb{R}^{n\times n}$, $\eta \mapsto P(\eta)$, is well defined by

$$A(\eta)^\top P(\eta) + P(\eta)\, A(\eta) = -I\;,\;\; P(\eta) > 0\;.$$

It is well known that for any fixed $q_0 \in \Omega$ the map $V(q_0,\;\cdot\;)\colon \mathbb{R}^n \to \mathbb{R}$, $\xi \mapsto \xi^\top P(q_0)\,\xi$, is a Lyapunov function of the system (2) on some neighborhood $D(q) \subseteq \mathbb{R}^n$ of 0 (see [7], proof of theorem 3.7). Therefore we call

$$V\colon \Omega \times \mathbb{R}^n \to \mathbb{R}\;,\quad (\eta, \xi) \mapsto \xi^\top P(\eta)\,\xi \qquad (3)$$

a *parametric Lyapunov function*. Keeping q_0 fixed, according to [7], theorem 3.1, a *region of attraction* $G(q_0) \subseteq \mathbb{R}^n$ can be constructed. Let

$$\mathcal{B}(R) := \{\xi|\,\xi \in \mathbb{R}^n,\, ||\xi||_2 \le R\} \qquad (4)$$

denote the closed ball with center at the origin and radius $R \in \mathbb{R}_0^+$. Then choose a radius $R(q_0) > 0$ by assumption (A2), such that the following inequality holds for all $\xi \in \mathcal{B}(R(q_0)),\ \xi \neq 0$:

$$2\, ||P(q_0)||_s\; ||f(q_0,\, \xi)||_2\; ||\xi||_2^{-1} < 1\;. \qquad (5)$$

Here $||\cdot||_s$ denotes the matrix norm induced by the Euclidean vector norm: $||L||_s := \sup_{||x||_2=1} ||L\,x||_2$. With

$$G(q_0) := \{\xi|\; V(q_0,\, \xi) \le R(q_0)^2\, \lambda_{\text{min}}[P(q_0)]\} \qquad (6)$$

the Lyapunov method yields for all $x_0 \in G(q_0)$:

$$T^+(q_0,\, x_0) = \mathbb{R}_0^+\;, \qquad (7)$$

$$\varphi(q_0,\, x_0,\, t) \in G(q_0) \quad \forall\, t \in \mathbb{R}_0^+\;, \qquad (8)$$

$$\lim_{t\to\infty} \varphi(q_0,\, x_0,\, t) = 0\;. \qquad (9)$$

Since $q \in \Omega$ is assumed to be unknown, we look for a set $G_\Omega \subseteq \mathbb{R}^n$, such that (7) and (9) hold for all $x_0 \in G_\Omega$ *and all* $q_0 \in \Omega$. A neighborhood G_Ω of 0 with the mentioned properties will be called a *robust region of attraction* of the system (2). Of course G_Ω is allowed to depend on Ω, but not on the actual parameter value $q \in \Omega$. Note that we do not demand G_Ω to be invariant. The construction of a robust region of attraction can be done by intersecting the sets $G(q)$ for all $q \in \Omega$. To become even more explicit, we intersect the largest balls contained in $G(q)$, as proposed in [5] and [6]:

$$r(q) :=\\ R(q)\,\lambda_{\text{max}}[P(q)]^{-1/2}\,\lambda_{\text{min}}[P(q)]^{1/2} \in \mathbb{R}^+\;, \quad (10)$$

$$G_\Omega := \mathcal{B}(\inf_{q\in\Omega} r(q))\;. \qquad (11)$$

Therefore computing G_Ω is a minimization problem. In order to avoid ending up with $G_\Omega = \{0\}$, one may assume $R\colon \Omega \to \mathbb{R}^+$ to be continuous, implying $r\colon \Omega \to \mathbb{R}^+$ to be continuous too. Furthermore restrictions on the derivates of $r(q)$ are to be stated. Since $r(\,\cdot\,)$ involves solving the Lyapunov equation and computation of eigenvalues, it seems impossible to verify such restrictions by given f and A. In practice one will simply 'solve' the minimization problem by evaluating $r(q)$ for an enormous number of parameter values $q \in \Omega$, still at the risk of missing the worst case.

In the following sections a method is established involving only a finite number of parameter values. By this we overcome the above kind of problems, and — even if $R(\,\cdot\,)$ happens to be discontinuous— a robust region of attraction can be computed in a reliable manner.

3. ARBITRARY KNOWN PARAMETER

In this section an arbitrary parameter $q_0 \in \Omega$ is fixed. We are interested in a cube $\mathcal{C}$ with center q_0 and positive radius $s \in \mathbb{R}^+$ such that the map $V(q_0\ \cdot)$ defined by (3) is a common Lyapunov function of system (2) for all $q \in \mathcal{C}$. Hereby the following definitions are used for shortness. Let $||\cdot||_\infty$ denote the maximum norm, i. e. $||\eta||_\infty := \max\{|\eta_1|,\ |\eta_2|,\ \ldots\ |\eta_m|\}$ for $\eta \in \mathbb{R}^m$. Then

$$\mathcal{C}(\eta,\, s) := \{v|\ v \in \mathbb{R}^m,\ ||v-\eta||_\infty \le s\} \qquad (12)$$

denotes the closed cube with center η and radius s. Further let

$$\mathcal{V}(\eta,\, s) := \{v|\ |v_i - \zeta_i| = s\ \ \forall\, 1 \le i \le m\} \qquad (13)$$

denote the vertices of $\mathcal{C}(\eta,\, s)$.

When as a first step the nonlinearity f is neglected, the following theorem presents a radius formula for the cube of interest.

Theorem 1. Let $q_0 \in \mathbb{R}^m$, an affine map $A\colon \mathbb{R}^m \to \mathbb{R}^{n\times n}$ and $\alpha \in [0,1)$ be given. Assume $A(q_0)$ to be stable. Let H denote the mapping from assumption (A3) and $P(q_0)$ the solution of the Lyapunov equation as in (P3). Let

$$\begin{aligned} s(\alpha,\, q_0) := \sup\{s|\ q \in \mathcal{C}(q_0,\, s)\ \ \Rightarrow \\ A(q)^\top P(q_0) + P(q_0)\, A(q) + \alpha\, I < 0\}\,. \end{aligned} \qquad (14)$$

Then the following equation holds:

$$\begin{aligned} s(\alpha,\, q_0) = (1-\alpha)\,(\max\{||H(\delta)^\top P(q_0) + \\ P(q_0)\, H(\delta)||_s|\ \delta \in \mathcal{V}(0,\,1)\})^{-1}. \end{aligned} \qquad (15)$$

The latter quotient is considered to be ∞ whenever the divisor happens to be 0.

Proof. For shortness let

$$\begin{aligned} M(d) &:= A(q_0+d)^\top P(q_0) + P(q_0)\, A(q_0+d) + \alpha\, I, \\ N(d) &:= H(d)^\top P(q_0) + P(q_0)\, H(d) \end{aligned}$$

for $d \in \mathbb{R}^m$, hence

$$M(d) = N(d) - (1-\alpha)\, I\,. \qquad (16)$$

Note that $N(\,\cdot\,)\colon \mathbb{R}^m \to \mathbb{R}^{n\times n}$ is linear. This implies that $\max\{||N(d)||_s|\ d \in \mathcal{V}(0,\,1)\}$ becomes zero if and only if $N(d) \equiv 0$. In this case, equation (14) yields $s(\alpha,\, q_0) = \infty$. Therefore only $\max\{||N(d)||_s|\ d \in \mathcal{V}(0,\,1)\} \ne 0$ has to be treated in the following. Now fix an arbitrary radius $s \in \mathbb{R}_0^+$ and claim:

$$\begin{aligned} ||N(d)||_s < 1-\alpha\ \forall\, d \in \mathcal{C}(0,\, s) \\ \Leftrightarrow \quad M(d) < 0\ \forall\, d \in \mathcal{C}(0,\, s)\,. \end{aligned} \qquad (17)$$

We prove (17) by first assuming the left hand side to hold. For an arbitrary d with $||N(d)||_s < 1-\alpha$ and an arbitrary $\xi \in \mathbb{R}^n \setminus \{0\}$ this yields

$$\begin{aligned} \xi^\top M(d)\, \xi &= \xi^\top N(d)\, \xi - (1-\alpha)\, ||\xi||_2^2 \\ &\le (||N(d)||_s - (1-\alpha))\, ||\xi||_2^2 \\ &< 0\,. \end{aligned} \qquad (18)$$

Therefore the right hand side of (17) is implied by the left hand side. Next assume the left hand side of (17) not to hold. Hence there exists a $d_0 \in \mathcal{C}(0,\, s)$ such that $||N(d_0)||_s \ge 1-\alpha$. From $N(\,\cdot\,)$ being linear we obtain $N(-d_0) = -N(d_0)$ and therefore $||N(-d_0)||_s = ||N(d_0)||_s \ge 1-\alpha$. From both $N(d_0)$ and $N(-d_0)$ being symmetric we observe $|\lambda_{\max}[N(-d_0)]| = |\lambda_{\max}[N(d_0)]| \ge 1-\alpha$. Hence one of the above eigenvalues is greater or equal $1-\alpha$. Without loss of generality we therefore assume $\lambda_{\max}[N(d_0)] \ge 1-\alpha$. For an according eigenvector $v \ne 0$ this yields:

$$\begin{aligned} v^\top M(d_0)\, v &= v^\top N(d_0)\, v - (1-\alpha)\, ||v||_2^2 \\ &= (\lambda_{\max}[N(d_0)] - (1-\alpha))\, ||v||_2^2 \\ &\ge 0\,. \end{aligned} \qquad (19)$$

Hence $M(d_0)$ is not negative definite and the right hand side of (17) is not fulfilled. This completes the prove of (17). We state another claim:

$$\begin{aligned} s < (1-\alpha)(\max_{d\in\mathcal{V}(0,\,1)} ||N(d)||_s)^{-1} \\ \Leftrightarrow \quad ||N(d)||_s < (1-\alpha) \quad \forall\, d \in \mathcal{C}(0,\, s)\,. \end{aligned} \qquad (20)$$

Note that the operator norm $||\cdot||_s\colon \mathbb{R}^{n\times n} \to \mathbb{R}_0^+$ is a convex map. Since $N(\,\cdot\,)$ is linear, $||N(\,\cdot\,)||_s\colon \mathbb{R}^m \to \mathbb{R}_0^+$ is convex too. Hence on convex hulls of a finite number of vertices $||N(\,\cdot\,)||_s$ becomes maximal for one of the vertices. This yields

$$\begin{aligned}\max_{d\in\mathcal{C}(0,\,s)} ||N(d)||_s &= s \max_{d\in\mathcal{C}(0,\,1)} ||N(d)||_s \\ &= s \max_{d\in\mathcal{V}(0,\,1)} ||N(d)||_s \,, \qquad (21)\end{aligned}$$

implying (20). From both (17) and (20) and by the substitution $q = q_0 + d$ we observe

$$\begin{aligned}& M(q-q_0) < 0 \quad \forall\, q \in \mathcal{C}(q_0,\, s) \\ \Leftrightarrow\quad & M(d) < 0 \quad \forall\, d \in \mathcal{C}(0,\, s) \\ \Leftrightarrow\quad & s < (1-\alpha)(\max_{d\in\mathcal{V}(0,\,1)} ||N(d)||_s)^{-1} \,. \qquad (22)\end{aligned}$$

Therefore the supremum of all s such that

$$M(q-q_0) < 0 \quad \forall\, q \in \mathcal{C}(q_0,\, s) \qquad (23)$$

holds is equal to

$$(1-\alpha)(\max_{d\in\mathcal{V}(0,\,1)} ||N(d)||_s)^{-1} \,. \qquad (24)$$

This completes the proof of the theorem. $\square$

In [8] theorem 2.6 the above is stated for the special case $\alpha = 0$. For a proof [4] is cited, where an algorithm is given for computing a common Lyapunov function. Thereby $s = s(0,\, q_0)$ is the maximal radius such that $V(q_0,\ \cdot\ )$ is a Lyapunov function of the linear system $\dot{x} = A(q)\,x$ for all $q \in \mathcal{C}(q_0,\, s)$. Choosing the newly introduced *design parameter* α to be positive clearly results in a smaller cube. The advantage of this is, that it allows to discard the nonlinear term f. The following corollary illustrates this idea.

Corollary 1. Let A, f, Ω, H, P, and V be given according to (A1) to (A15), (P3) and (3). For an arbitrary but fixed $\alpha \in (0,1)$ and $q_0 \in \Omega$ let

$$\Omega(\alpha,\, q_0) := \mathcal{C}^\circ(q_0,\, s(\alpha,\, q_0)) \,, \qquad (25)$$

where $\mathcal{C}^\circ$ denotes the inner of the set $\mathcal{C}$. From (A2) choose a radius $R(\alpha,\, q_0) > 0$:

$$R(\alpha,\, q_0) := \delta(\Omega(\alpha,\, q_0),\, \tfrac{1}{2}\,\alpha\, ||P(q_0)||_s^{-1}) \,. \qquad (26)$$

Further let

$$\begin{aligned}& D(\alpha,\, q_0) := \mathcal{B}(R(\alpha,\, q_0)) \,, \qquad (27) \\ & G(\alpha,\, q_0) := \\ & \quad \{\xi |\ V(q_0,\, \xi) \le R(\alpha,\, q_0)^2\, \lambda_{\min}[P(q_0)]\} \,. \qquad (28)\end{aligned}$$

Then for all $q \in \Omega(\alpha,\, q_0)$ the following holds:

(i) $V(q_0,\ \cdot\ )$ is a Lyapunov function on $D(\alpha,\, q_0)$ of the system (2).

(ii) $G(\alpha,\, q_0)$ is a invariant region of attraction of the system (2).

Proof. Since $V(q_0,\ \cdot\ )$ is quadratic, we only need to show that for all $q \in \Omega(\alpha,\, q_0)$, $x_0 \in \mathbb{R}^n$, $t \in T^+(q,\, x_0)$ and $\varphi(q,\, x_0,\, t) \in D(\alpha,\, q_0) \setminus \{0\}$

$$\dot{V}(q_0,\, \varphi(q,\, x_0,\, t)) < 0 \qquad (29)$$

holds. By equation (26) observe

$$2\, ||P(q_0)||_s\, ||f(q,\, \xi)||_2\, ||\xi||_2^{-1} \le \alpha \qquad (30)$$

for all $q \in \Omega(\alpha,\, q_0)$ and all $\xi \in D(\alpha,\, q_0) \setminus \{0\}$. Fix arbitrary q, x_0 and t from the above and denote $\varphi(q,\, x_0,\, t)$ by x. Then the definition of $s(\alpha,\, q_0)$ in theorem 1 yields

$$\begin{aligned}& \dot{V}(q_0,\, x) \\ &= \dot{x}^\top P(q_0)\, x + x^\top P(q_0)\, \dot{x} \\ &= x^\top [A(q)^\top P(q_0) + P(q_0)\, A(q)]\, x \\ &\quad + 2\, x^\top P(q_0)\, f(q,\, x) \\ &< -\alpha\, ||x||_2^2 + 2\, ||x||_2\, ||P(q_0)||_s\, ||f(q,\, x)||_2 \\ &= -\, ||x||_2^2\, (\alpha - 2\, ||x||_2^{-1} ||P(q_0)||_s\, ||f(q,\, x)||_2) \\ &\le 0 \,. \qquad (31)\end{aligned}$$

This yields (i). For any $\xi \in G(\alpha,\, q_0)$ we get

$$\begin{aligned} R(\alpha,\, q_0)^2\, \lambda_{\min}[P(q_0)] &\ge \xi^\top P(q_0)\, \xi \\ &\ge ||\xi||_2^2\, \lambda_{\min}[P(q_0)] \,.\end{aligned}$$

And therefore $\xi \in D(\alpha,\, q_0)$, hence $G(\alpha,\, q_0) \subseteq D(\alpha,\, q_0)$. Now (ii) is obtained by the Lyapunov method. See theorem 3.1 and 3.7 in [7]. $\square$

Remark: From equation (15) we know $\Omega(\alpha,\, q_0) \searrow \{q_0\}$ when $\alpha \nearrow 1$. When $\alpha_1 > \alpha_2$ in general $R(\alpha_1,\, q_0) \ge R(\alpha_2,\, q_0)$ is expected. Therefore α near 1 is likely to yield a large region of attraction valid for a small set of parameters $\Omega(\alpha,\, q_0)$ —and the contrary happens when α near 0.

4. UNCERTAIN PARAMETER

Transferring the results of the previous section to the situation of an uncertain parameter is straight forward. We first assume a finite set of parameter values $q_1,\ \ldots\ ,\ q_N \in \Omega,\ N \in \mathbb{N}$, to be known, such that the union of the sets $\Omega(\alpha,\, q_i),\ 1 \le i \le N$, is a superset of Ω. Thereby $\alpha \in (0,\, 1)$ is chosen arbitrarily but fixed. From corollary 1 we construct a piecewise constant parametric Lyapunov function: Therefore let

$$\begin{aligned} Z_1 &:= \Omega(\alpha,\, q_1) \,, \qquad (32) \\ Z_i &:= \Omega(\alpha,\, q_i) \setminus [\bigcup_{1\le j<i} Z_j] \,, \qquad (33)\end{aligned}$$

and

$$\begin{aligned} V_{\mathrm{pc}}(\alpha,\, Q,\, \eta,\, \xi) &:= \sum_{1\le i\le N} \chi_{Z_i}(\eta)\, V(q_i,\, \xi) \,, \qquad (34) \\ D_\Omega(\alpha,\, Q) &:= \mathcal{B}(\min_{1\le i\le N} R(\alpha,\, q_i)) \,. \qquad (35)\end{aligned}$$

Here χ_Z denotes the characteristic function of a set Z, that is $\chi_Z(\eta) = 1$ for all $\eta \in Z$ and $\chi_Z(\eta) = 0$ far all $\eta \notin Z$. Corollary 1 yields that $V_{\text{pc}}(\alpha, Q, q, \cdot)$ is a Lyapunov function of system (2) on the domain $D_\Omega(\alpha, Q)$. Analogous to equation (10)

$$r(\alpha, q_i) := R(\alpha, q_i)\, \lambda_{\max}[P(q_i)]^{-1/2}\, \lambda_{\min}[P(q_i)]^{1/2} \tag{36}$$

is the radius of the largest ball within $G(\alpha, q_i)$. Let

$$G_\Omega(\alpha, Q) := \mathcal{B}(\min_{1\le i\le N} r(\alpha, q_i)) \,. \tag{37}$$

Note that the above minimum is positive since the $r(\alpha, q_i)$ are. Observe that

$$G^\star_\Omega(\alpha, Q) := \mathcal{B}(\max_{1\le i\le N} R(\alpha, q_i)) \tag{38}$$

is a superset of $G(\alpha, q_i)$ for all i. Hence by corollary 1 it follows for all $q \in \Omega$ and all $x_0 \in G_\Omega(\alpha, Q)$:

$$T^+(q, x_0) = \mathbb{R}_0^+ \,, \tag{39}$$

$$\varphi(q, x_0, t) \in G^\star_\Omega(\alpha, Q) \quad \forall\, t \in \mathbb{R}_0^+ \,, \tag{40}$$

$$\lim_{t\to\infty} \varphi(q, x_0, t) = 0 \,. \tag{41}$$

This proves $G(\alpha, Q)$ to be a robust region of attraction of system (2). The following theorem treats the question on how to construct a finite set of parameter values Q such that the above can be applied.

Theorem 2. Fix some $\alpha \in (0,1)$ and let $(q_i)_{i\in\mathbb{N}}$ be an infinite sequence of parameter values in Ω. Assume the set $Q = \{q_i |\, i \in \mathbb{N}\}$ to be dense in Ω. Then there exists an $N \in \mathbb{N}$ such that

$$\bigcup_{1\le i\le N} \Omega(\alpha, q_i) \supseteq \Omega \,. \tag{42}$$

Proof. Let $C_i := \mathcal{C}^\circ(q_i, s(\alpha, q_i))$ for all $i \in \mathbb{N}$. Observe from eq. (15) that the map $s(\alpha, \cdot)$ is continuous on Ω. By assumption (A5) Ω is compact. This implies $s(\alpha, \eta_0) = \inf_{\eta\in\Omega} s(\alpha, \eta) := s_{\min}(\alpha)$ for some $\eta_0 \in \Omega$. Observe $s(\alpha, \cdot)$ to be positive on Ω, hence $s_{\min}(\alpha) > 0$. Now fix an arbitrary $q \in \Omega$. From Q being dense in Ω we know a $q_i \in \Omega$ to exist, such that $||q - q_i||_\infty < s_{\min}(\alpha)$. Therefore $\cup_{i\in\mathbb{N}} C_i \supseteq \Omega$. Again from Ω being compact we can choose a finite number of sets $C_{i_1}, \ldots C_{i_J}$, $J \in \mathbb{N}$, such that $\cup_{j\le J} C_{i_j} \supseteq \Omega$. This completes the proof. □

Whenever a countable dense set of parameter values is known, by the above theorem an algorithm can be set up in order to determine a suitable finite set of parameter values. For the case of Ω being a polyrectangular with dimension m, we present such an algorithm in explicit form. Choose a fix design parameter $\alpha \in (0, 1)$ and run algorithm S:

(S1) Assign $i \leftarrow 0$.

(S2) Assign $i \leftarrow i + 1$ and $Q \leftarrow \frac{1}{2^i}\mathbb{Z}^m \cap \Omega$, where $\frac{1}{2^i}\mathbb{Z}^m := \{\frac{1}{2^i} z |\, z \in \mathbb{Z}^m\}$ denotes the rectangular grid with distance $\frac{1}{2^i}$ between the grid points along the axis.

(S3) If $Q = \emptyset$ go to step (S2).

(S4) If $\min_{\eta\in Q} s(\alpha, \eta) \le 2^{-i}$ go to step (S2) else terminate.

The algorithm terminates after a finite number of steps, because: (i) Assuming the polyrectangular Ω to be of dimension m implies the condition in (S3) to be not fulfilled for all i greater than a certain $i_0 \in \mathbb{N}$. (ii) From the proof of theorem 2 $s(\alpha, \eta)$ is known to be greater than $s_{\min}(\alpha) > 0$ for all $\eta \in \Omega$. Hence for certain $i_1 \in \mathbb{N}_0$ the condition in (S4) is not fulfilled. After having terminated, clearly Q holds a finite number of parameter values such that $\cup_{\eta\in Q}\Omega(\alpha, \eta)$ is a superset of Ω.

5. EXAMPLE

Consider the uncertain second order system given by the differential equation (2) when $n = m = 2$ and

$$A\left(\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}\right) := \begin{bmatrix} -q_1 & 1 \\ 0 & -q_2 \end{bmatrix} , \tag{43}$$

$$f\left(\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) := \begin{bmatrix} 0 \\ (q_1 - 1)(x_1 x_2)^3 \end{bmatrix} . \tag{44}$$

As the uncertainty bounding set we choose

$$\Omega = [1, 2] \times [1, 2] \subset \mathbb{R}^2 \,. \tag{45}$$

Then for all $x,\ q \in \mathbb{R}^2$

$$||f(q, x)||_2 \le \frac{|q_1 - 1|}{8}(||x||_2)^6 \tag{46}$$

holds. Of course, finding a suitable boundary for the nonlinear term f remains a crucial task and has to be done individually for every application. Now set up δ according to assumption (A2) by

$$\delta(\mathcal{C}(q, s), \varepsilon) := \sqrt[5]{8\,\varepsilon\,(q_1 + s - 1)^{-1}} \,, \tag{47}$$

where $q \in \Omega$, $s \in \mathbb{R}^+$. Clearly assumptions (A1) to (A5) are fulfilled by the above. Therefore running algorithm S for any fixed $\alpha \in (0, 1)$ yields a finite set Q of parameter values. The number of parameter values (denoted by $|Q|$) is listed in table 1 for $\alpha \in \{0.3, 0.5, 0.7, 0.9\}$.

Table 1. Results of algoritm S.

α	0.3	0.5	0.7	0.9
$\lvert Q\rvert$	25	25	81	1089
r	0.709	0.786	0.840	0.884

To establish a region of attraction set $R(\,\cdot\,)$ as in equation (26):

$$R\,(\alpha,\,q) := \sqrt[5]{4\,\alpha\,(q_1 + s(\alpha,\,q) - 1)^{-1}(||P(q)||_s)^{-1}} \tag{48}$$

for $\alpha \in (0,\,1)$ and $q \in \Omega$. Computing $r(\alpha,\,q)$ is then done by applying the formula given in section 4. The minimum r of the radii $r(\alpha,\,q)$ over all q in Q then represents $G(\alpha,\,Q)$, which is a robust region of attraction. Again see table 1 for numerical results. The dark grey ball in fig. 1 represents $G(\alpha,\,Q)$ at $\alpha = 0.9$. The light grey area is an estimation of the actual robust region of attraction, established by numericly solving the differential equation for 1600 initial conditions and 25 parameter values each.

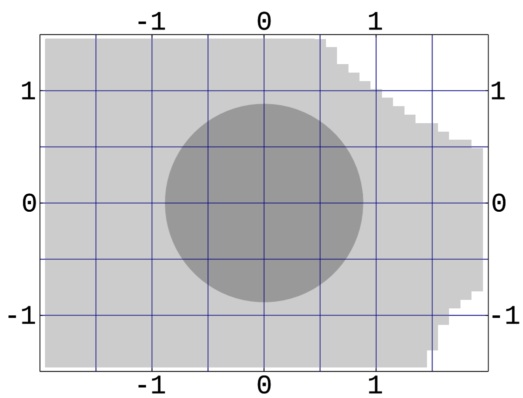

Fig. 1. Robust region of attraction.

6. CONCLUSIONS

Since the proposed method is based on discarding the nonlinear term f, a restrictive robust region of attraction is expected. At least when the nonlinearity has relevant influence on the dynamics within the actual region of attraction. In those cases it will be more appropriate to find a Lyapunov function corresponding to the nonlinearity's structure. However, often such a Lyapunov function is not known, hence there is no alternative to shaping the nonlinearity.

The construction of the robust region of attraction $G_\Omega(\alpha,\,Q)$ is done very much like G_Ω in [5], roughly stated in section 2 for easy reference. From approximating a parameter dependent Lyapunov function by a piecewise constant one, as a main advantage intersecting a infinite number of sets is avoided. Therefore we achieve a reliable result by a finite procedure. When choosing the design parameter α near 1, an enormous number of parameter values Q is expected to be computed, ending up in $G_\Omega(\alpha,\,Q) \approx G_\Omega$.

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